

Worcester County Mathematics League

Varsity Meet 4 – April 8, 2026

COACHES' COPY ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 4 - April 8, 2026
Answer Key



Round 1 - Elementary Number Theory

1. 33 or 33_8
2. 36
3. 252

Round 2 - Algebra I

1. $\frac{3}{2}$ or $1\frac{1}{2}$ or 1.5
2. $4 + 2\sqrt{14}$ or $2\sqrt{14} + 4$
3. 53

Round 3 - Geometry

1. 1 : 4
2. $3\sqrt{5}$
3. $\frac{16\sqrt{5}}{5}$

Round 4 - Logs, Exponents, and Radicals

1. 4
2. $-\frac{31}{13}$ or $-2\frac{5}{13}$ or $-2.\overline{384615}$
3. $\frac{1 + \sqrt{5}}{2}$ or $\frac{1}{2} + \frac{\sqrt{5}}{2}$, sums in either order.

Round 5 - Trigonometry

1. $15\sqrt{2}$
2. (1, 1, 0) exact order
3. $(75^\circ, 4)$ or $(75, 4)$ exact order

Team Round

1. 5
2. 11
3. $50\pi - 100$
4. $73 + 12\sqrt{2}$ or $12\sqrt{2} + 73$
5. $\frac{7}{11}$
6. 16
7. -5
8. (2, 1) and $(-2, -1)$ need both ordered pairs, either ordered pair first
9. $-2x\sqrt{1 - x^2}$

Worcester County Mathematics League
Varsity Meet 4 - April 8, 2026
Round 1 - Elementary Number Theory



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Represent 11011_2 in base 8.
2. Find the least common multiple of all whole numbers less than 25 that have the same number of factors as 25.
3. Find the number of trailing zero's in the base 2025 representation of $2025!$ (2025 factorial).

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

Tantasqua, Quaboag, QSC

Worcester County Mathematics League
Varsity Meet 4 - April 8, 2026
Round 2 - Algebra I



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Find the slope of the line (as graphed on the x - y plane) defined by the equation $2y - 3x - 4 = 0$.

2. Find the sum of the absolute value of all solutions to the following equation:

$$|9 - x^2| = 5$$

3. Let S be a subset of the Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ with 7 members where no two pairs of numbers in S have the same sum. Let K be the sum of the seven members of S . Find the smallest possible value of K .

ANSWERS

(1 pt) 1. _____

(2 pts) 2. _____

(3 pts) 3. _____

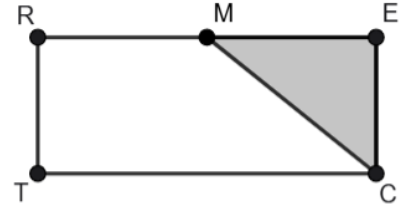
Oxford, Shrewsbury, Mass. Academy/QSC



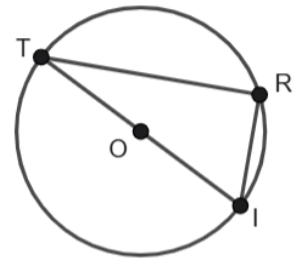
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NO CALCULATORS ALLOWED

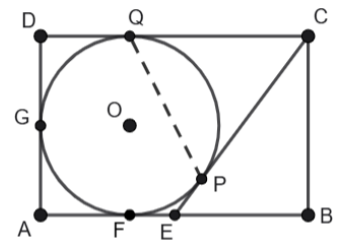
1. Given rectangle $RECT$ shown in the figure on the right, where M is the midpoint of $\overline{RE}$: find the ratio $m : n$ of the area of $\triangle MEC$ (shaded) to the area of $RECT$. Your answer must be in simplest terms, that is, m and n must be integers with no common factors.



2. Triangle $\triangle TRI$ is inscribed in circle O such that IT is a diameter of the circle, as shown on the right. The area of $\triangle TRI$ is 45, $IT = 15$ and $RI < TR$. Find RI .



3. Given circle O tangent to three sides of rectangle $ABCD$ at points Q , F , and G ; $\overline{CE}$ tangent to circle O at P ; and E lying on $\overline{AB}$, shown at right: if $BE = 6$ and $BC = 8$, find PQ . Express your answer in simplest radical form.



ANSWERS

(1 pt) 1. _____ : _____

(2 pts) 2. _____

(3 pts) 3. $PQ =$ _____

Worcester County Mathematics League
Varsity Meet 4 - April 8, 2026
Round 4 - Logs, Exponents, and Radicals



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Find n given that n is an integer and

$$n < \log_2 25 < n + 1$$

2. Find x :

$$\left(\frac{1}{81}\right)^{2x-1} = (243)^{x+7}$$

3. Find x if

$$x = \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}$$

ANSWERS

(1 pt) 1. $n =$ _____

(2 pts) 2. $x =$ _____

(3 pts) 3. $x =$ _____

Tantasqua, Tantasqua, QSC

Worcester County Mathematics League
Varsity Meet 4 - April 8, 2026
Round 5 - Trigonometry



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Find the area of $\triangle ABC$ if $m\angle A = 45^\circ$, $AB = 6$, and $AC = 10$.

2. Let $\frac{\frac{1}{\cos x}}{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}} = a \sin^b x \cos^c x$. Find the ordered triple (a, b, c) .

3. Let y^* be the maximum value of $f(\theta)$ and let θ^* be the smallest positive value of θ such that $f(\theta) = y^*$ where:

$$f(\theta) = 2\sqrt{3} \sin(\theta + 45^\circ) + 2 \sin(\theta - 45^\circ).$$

Find the ordered pair (θ^*, y^*) where θ^* is measured in degrees.

ANSWERS

(1 pt) 1. _____

(2 pts) 2. $(a, b, c) = \left(\rule{1.5cm}{0.4pt} \right)$

(3 pts) 3. $(\theta^*, y^*) = \left(\rule{1.5cm}{0.4pt} \right)$

South, Athol, QSC

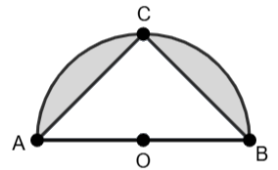
Worcester County Mathematics League
Varsity Meet 4 - April 8, 2026
Team Round



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Let N equal the product of the first 100 odd prime numbers. Find the least significant (right most) digit of N , as written in base 10.
2. A battalion of 880 soldiers is made up of four companies of equal size. Each of those companies is made up of five platoons of equal size. Each of those platoons is made up of four squads of equal size. Find the number of soldiers in a squad.
3. Isosceles $\triangle ABC$ is inscribed in semicircle O as shown in the figure at right. The radius of the semicircle $OA = 10$. Find the exact area of the shaded region. Do not approximate π .



4. Given that $\log_{\sqrt{n}} m^3 = \sqrt{2}$, find:

$$[\log_m (n^2 \cdot m)]^2$$

5. In triangle $\triangle ABC$, $\tan \angle A = 3$ and $\tan \angle B = 4$. Find the value of $\tan \angle C$.
6. The product of a set of distinct positive integers is 126. What is the least possible sum of this set of integers?
7. Given that $x + y = 4$ and $x + z = 9$, find the value of $y - z$.
8. Given that $(a + bi)^2 = 3 + 4i$, where a and b are Real numbers and $i = \sqrt{-1}$, find all possible ordered pairs (a, b) .

9. Given the following expression:

$$\sin [2 \cos^{-1}(-x)]$$

find an equivalent expression in terms of x that has no trigonometric functions.

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Answer Key



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Round 4 - Logs, Exponents, and Radicals

1. 4
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Round 5 - Trigonometry

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8. (2, 1) and $(-2, -1)$ need both ordered pairs, either ordered pair first
9. $-2x\sqrt{1 - x^2}$

Round 1 - Elementary Number Theory

1. Represent 11011_2 in base 8.

Solution: Let $n = 11011_2$. Recall that a number represented base 2 can be converted to base 10 by multiplying its digits (0 or 1) by decreasing powers of 2, starting with 2^{k-1} given k digits in the base 2 representation. This method also works for base 8 numbers using powers of 8. In this case n has 5 binary (base 2) digits, and:

$$\begin{aligned} n &= 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 1 \cdot 16 + 1 \cdot 8 + 0 + 1 \cdot 2 + 1 \cdot 1 \\ &= 16 + 8 + 2 + 1 = 27. \end{aligned}$$

Consider that the leading, leftmost digit of the base 8 representation of n is determined by the highest power of 8 less than n . Note that $8^2 = 64 > 27$. Therefore the base 8 representation of n has only two digits, d_1 and d_0 . Divide 27 by 8: $27 \div 8 = 3 = d_1$ with a remainder of $3 = d_0$. Therefore $n = 3 \cdot 8 + 3$ and $n = \boxed{33_8}$.

Alternative solution: Divide $n = 11011$ into groups of three binary digits starting from the rightmost digits, preserving the order: 11 011. Note that moving three binary digits right to left corresponds to an increase in the value of a digit by a factor of 8 because $2^3 = 8$. Thus, each group of three base 2 digits corresponds to a single base 8 digit. Then $011_2 = 0 \cdot 2^2 + 1 \cdot 2 + 1 = 2 + 1 = 3$. Therefore the two base 8 digits of n are both 3 and $n = 33_8$, as above.

2. Find the least common multiple of all whole numbers less than 25 that have the same number of factors as 25.

Solution: Note that 25 has three factors 1, 5 and 25, and that every square of a prime number p has three factors 1, p and p^2 . Also, note that prime numbers have two factors and composite numbers (other than perfect squares of prime numbers) have at least four factors. Therefore only squares of prime numbers have three factors.

Note that 2 and 3 are the only prime numbers less than 5, so $2^2 = 4$ and $3^2 = 9$ are the only squares of primes less than 25. The least common multiple of 4 and 9 is $4 \cdot 9 = \boxed{36}$. Note that $4^2 = 16$ is a perfect square but not a square of a prime. It has five factors (1, 2, 4, 8, 16), not three.

3. Find the number of trailing zero's in the base 2025 representation of 2025! (2025 factorial).

Solution: Let $N = 2025!$, and let n be the number of trailing zeros in the base 2025 representation of N . Then n is equal to the number of factors of 2025 in N , just as the number of trailing zeros in the base 10 representation of N equals the number of factors of 10 in N . Note that $2025 = 45^2 = (9 \cdot 5)^2 = (3^2 \cdot 5)^2 = 3^4 \cdot 5^2$. Let n_3 and n_5 be the number of factors of 3 and 5, respectively, in N . Then $N = K \cdot 3^{n_3} \cdot 5^{n_5}$, where K is equal to the product of the powers of all prime factors of N other than 3 and 5. Then n is equal to the lesser of $\lfloor \frac{n_3}{4} \rfloor$ and $\lfloor \frac{n_5}{2} \rfloor$, where $\lfloor x \rfloor$ is the “floor” function that rounds x down to the largest integer less than x .

It remains to find n_3 and n_5 . Let set $S = \{1, 2, 3, \dots, 2025\}$ so that N is the product of the elements of S . Let t_k be the number of elements of S that have 3^k as a factor. Then $n_3 = t_1 + t_2 + \dots + t_6$. That is, count the factors of 3 by counting the elements of S that have 3 as a factor, adding those that have 3^2 as a factor, and so on up to $3^6 = 9^3 = 27^2 = 729$. There are no factors of $3^7 = 2187$ in any element of S because $2187 > 2025$.

Note that 3^4 divides $2025 = 3^4 \cdot 5^2$, and that t_1, t_2, t_3 and t_4 can be found by simply dividing 2025 by the corresponding power of 3. Calculate: $2025 \div 3 = 675 = t_1$, $2025 \div 3^2 = 675 \div 3 = 225 = t_2$, $2025 \div 3^3 = 225 \div 3 = 75 = t_3$, and $2025 \div 3^4 = 75 \div 3 = 25 = t_4$. Then $t_5 = \lfloor 2025 \div 3^5 \rfloor = \lfloor 25 \div 3 \rfloor = 8$ and $t_6 = \lfloor 2025 \div 3^6 \rfloor = \lfloor \lfloor 25 \div 3 \rfloor \div 3 \rfloor = \lfloor 8 \div 3 \rfloor = 2$. Check that there are exactly two elements of S with factors of 3^6 : 729 and $2 \cdot 729 = 1458$. Sum t_1 through t_6 : $n_3 = 675 + 225 + 75 + 25 + 8 + 2 = 900 + 100 + 10 = 1010$.

Calculate n_5 in a similar manner. Let v_k be the number of elements of S that have 5^k as a factor. Note that $5^5 = 5 \cdot 5^4 = 5 \cdot (5^2)^2 = 5 \cdot 25^2 = 5 \cdot 625 = 3125 > 2025$ and $v_k = 0$ for $k > 4$. Thus, $n_5 = v_1 + v_2 + v_3 + v_4$. Calculate $v_1 = 2025 \div 5 = 405$, $v_2 = 2025 \div 5^2 = 405 \div 5 = 81$, $v_3 = \lfloor 2025 \div 5^3 \rfloor = \lfloor 81 \div 5 \rfloor = 16$, and $v_4 = \lfloor 2025 \div 5^4 \rfloor = \lfloor 81 \div 5^2 \rfloor = \lfloor \lfloor 81 \div 5 \rfloor \div 5 \rfloor = \lfloor 16 \div 5 \rfloor = 3$. Sum v_1 through v_4 : $n_5 = 405 + 81 + 16 + 3 = 505$.

Finally, $\lfloor \frac{n_3}{4} \rfloor = \lfloor \frac{1010}{4} \rfloor = 252$ and $\lfloor \frac{n_5}{2} \rfloor = \lfloor \frac{505}{2} \rfloor = 252$. The lesser of 252 and 252 is of course 252, so $N = \boxed{252}$.

Round 2 - Algebra I

1. Find the slope of the line (as graphed on the x - y plane) defined by the equation $2y - 3x - 4 = 0$. Express your answer in simplest form $\frac{m}{n}$.

Solution: Recall the slope-intercept form of the equation of a line: $y = mx + b$. Here m is the slope and b is the y -intercept of the line. Add $3x + 4$ to both sides of the equation, then divide both sides by 2:

$$\begin{aligned} 2y - 3x - 4 &= 0 \\ 2y &= 3x + 4 \\ y &= \frac{3x + 4}{2} = \frac{3}{2}x + \frac{4}{2}, \end{aligned}$$

and the equation is in slope-intercept form. Equate the right hand side to $mx + b$ and $m = \boxed{\frac{3}{2}}$.

2. Find the sum of the absolute value of all solutions to the following equation:

$$|9 - x^2| = 5.$$

Solution: Recall that if $|y| = z$, then $y = z$ or $y = -z$. Therefore there are two equations to solve: $9 - x^2 = 5$ and $9 - x^2 = -5$.

Solve the first equation: $9 - 5 = 4 = x^2$ and $x = \pm\sqrt{4} = \pm 2$. Solve the second equation: $9 + 5 = 14 = x^2$ and $x = \pm\sqrt{14}$.

Add the absolute value of the solutions and $|2| + |-2| + |\sqrt{14}| + |-\sqrt{14}| = 2 + 2 + \sqrt{14} + \sqrt{14} = \boxed{4 + 2\sqrt{14}}$.

3. Let S be a subset of the Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ with 7 members where no two pairs of numbers in S have the same sum. Let K be the sum of the seven members of S . Find the smallest possible value of K .

Solution: Define S_n as the subset with n elements where no two pairs have the same sum, and the sum of whose elements is the least among all subsets with n elements each of whose pairs have a distinct sum. Name elements such that $S_n = \{k_1, k_2, \dots, k_n\}$ in increasing order, k_n being the largest element. Start with $S_3 = \{k_1, k_2, k_3\} = \{1, 2, 3\}$ (the three smallest Natural numbers), and add numbers one at a time to find S_4, S_5 , etc. Note that the sums for S_3 are $1 + 2, 1 + 3$, and $2 + 3$, or 3, 4 and 5, so that $S_3 = \{1, 2, 3\}$ satisfies the pair sum property. It also has the smallest sum among all three element subsets of $\mathbb{N}$.

Next, note that $k_4 \neq 4$ because $1 + 4 = 2 + 3$. If $k_4 = 5$, then the sums are the previous sums 3, 4, 5 and 5 added to each of the elements in S_3 , or $5 + 1, 5 + 2, 5 + 3$. The sums for S_4 are therefore 3, 4, 5, 6, 7, 8 and $S_4 = \{1, 2, 3, 5\}$.

Next, note that $k_5 \neq 6$ because $6 + 1 = 7 = 5 + 2$. Likewise, $7 + 1 = 5 + 3$, so $k_5 \neq 7$. If $k_5 = 8$, then 8 pairs up with the members of S_4 to create the new sums $1 + 8, 2 + 8, 3 + 8, 5 + 8$, and the full set of sums for S_5 is $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 13\}$. Thus $S_5 = \{1, 2, 3, 5, 8\}$.

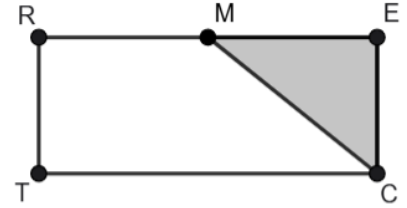
Note that $k_6 \neq 9$ because $9 + 1 = 8 + 2$. Likewise, $k_6 \neq 10$ because $10 + 1 = 8 + 3$, $k_6 \neq 11$ because $11 + 2 = 8 + 5$, and $k_6 \neq 12$ because $12 + 1 = 8 + 5$. If $k_6 = 13$, then the new sums are $1 + 13, 2 + 13, 3 + 13, 5 + 13, 8 + 13$ and the set of sums for S_6 is $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 21\}$. Thus, $S_6 = \{1, 2, 3, 5, 8, 13\}$.

Note that $k_7 \neq 14$ because $14 + 1 = 13 + 2$; $k_7 \neq 15$ because $15 + 1 = 13 + 3$; $k_7 \neq 16$ and $k_7 \neq 17$ because $16 + 2 = 13 + 5 = 17 + 1$; $k_7 \neq 18, k_7 \neq 19$ and $k_7 \neq 20$ because $18 + 3 = 13 + 8 = 19 + 2 = 20 + 1$. If $k_7 = 21$ then the new sums are $1 + 21, 2 + 21, 3 + 21, 5 + 21, 8 + 21, 13 + 21$ and the set of sums for S_7 is $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 21, 22, 23, 24, 26, 29, 34\}$. Thus, $S_7 = \{1, 2, 3, 5, 8, 13, 21\}$. The sum of the elements in S_7 are $1 + 2 + 3 + 5 + 8 + 13 + 21 = \boxed{53}$.

Note that the elements of S_7 are Fibonacci numbers. It is not true, in general, that the set of elements with the smallest sum and all distinct sums of pairs always consists of Fibonacci numbers. In fact, $S_8 = \{1, 2, 3, 5, 8, 13, 21, 30\}$ ($k_8 \neq 34$) with distinct sums of pairs $\{31, 32, 33, 35, 38, 43, 51\}$ added to the set of sums of pairs of S_7 .

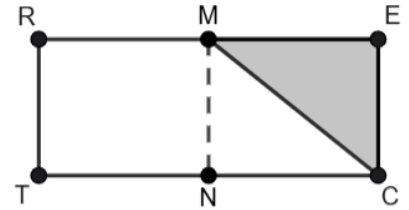
Round 3 - Geometry

1. Given rectangle $RECT$ shown in the figure on the right, where M is the midpoint of $\overline{RE}$: find the ratio $m : n$ of the area of $\triangle MEC$ (shaded) to the area of $RECT$. Your answer must be in simplest terms, that is, m and n must be integers with no common factors.



Solution: Denote the area of $RECT$ as $[RECT]$ and the area of

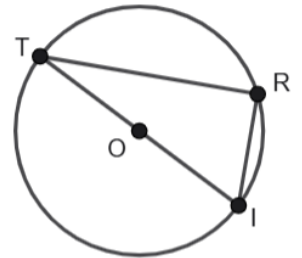
$\triangle MEC$ as $[\triangle MEC]$. Locate the midpoint of $\overline{CT}$, name it N , and construct the line segment $\overline{MN}$, as shown at right. Note that $\overline{MN}$ divides $RECT$ into two congruent rectangles $RMNT$ and $MECN$, each of which has half the area of $RECT$. Thus, the ratio of their areas is one half, or $[MECN] : [RECT] = 1 : 2$.



Next, note that diagonal $\overline{MC}$ divides square $MECN$ into congruent right triangles, so that $[\triangle MEC] : [MECN] = 1 : 2$. Finally,

$$\begin{aligned} [\triangle MEC] : [RECT] &= ([\triangle MEC] : [MECN]) \cdot ([MECN] : [RECT]) \\ &= (1 : 2) \cdot (1 : 2) = \boxed{1 : 4}. \end{aligned}$$

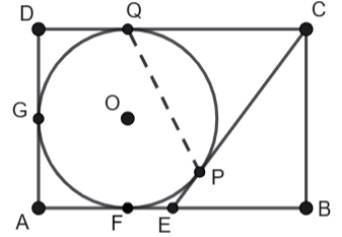
2. Triangle $\triangle TRI$ is inscribed in circle O such that IT is a diameter of the circle, as shown on the right. The area of $\triangle TRI$ is 45, $IT = 15$ and $RI < TR$. Find RI .



Solution: Note that $\triangle TRI$ is inscribed in a semicircle because diameter $\overline{IT}$ divides circle O into two semicircles. Recall that the measure of an inscribed angle is equal to half the measure of its intercepted arc. In this case, $\angle TRI$ intercepts a semicircle and half of 180° is 90° . Therefore $m\angle R = 90^\circ$ and $\triangle TRI$ is a right triangle.

Let $TR = a$, $RI = b$ and $TI = c$. Recall that the area of a right triangle is half the product of its two legs. For $\triangle TRI$, $45 = \frac{1}{2}ab$ and $ab = 90$. Recall the Pythagorean Theorem: $a^2 + b^2 = c^2 = 15^2 = 225$. Then $(a + b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2(90) = 225 + 180 = 405$ and $a + b = \sqrt{405} = \sqrt{81 \cdot 5} = 9\sqrt{5}$. Also, $(a - b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2(90) = 225 - 180 = 45$ and $a - b = \sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}$. Sum these two equations and $2a = 9\sqrt{5} + 3\sqrt{5} = 12\sqrt{5}$, so $a = 6\sqrt{5}$. Rewrite the $a - b$ equation and $RI = b = a - 3\sqrt{5} = 6\sqrt{5} - 3\sqrt{5} = \boxed{3\sqrt{5}}$.

3. Given circle O tangent to three sides of rectangle $ABCD$ at points Q , F , and G ; $\overline{CE}$ tangent to circle O at P ; and E lying on $\overline{AB}$, shown at right: if $BE = 6$ and $BC = 8$, find PQ . Express your answer in simplest radical form.



Solution: Note that $\angle EBC$ is a right angle and $BE : BC = 6 : 8 = 3 : 4$. Therefore $\triangle EBC$ is a $3 : 4 : 5$ right triangle with scale factor 2 and $CE = 2 \cdot 5 = 10$. Let $CQ = y$ and $EP = z$. Recall that tangent segments that are drawn from a common point to a circle are congruent. Then $\overline{CQ} \cong \overline{CP}$ and $\overline{EP} \cong \overline{EF}$, so that $CQ = CP = y$ and $EP = EF = z$.

Draw diameter $\overline{QF}$, as shown at right. Construct $\overline{PH}$ parallel to $\overline{AB}$ and perpendicular to $\overline{QF}$. Construct $\overline{PJ}$ parallel to $\overline{BC}$ and perpendicular to $\overline{AB}$. Note that $CP + EP = CE$ and

$$y + z = 10.$$

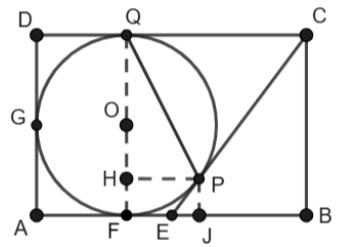
Also note that $BF = CQ = y$ because $BFQC$ is a rectangle. In addition, $BE + EF = BF$ and $6 + z = y$, or

$$y - z = 6$$

Add the two linear equations in y and z : $y + y + z - z = 10 + 6$, or $2y = 16$, so $y = 8$. Also $z = 10 - y = 10 - 8 = 2$.

Observe that $\triangle PHQ$ is a right triangle and that PQ can be calculated using the Pythagorean Theorem given PH and HQ . Also note that $\triangle PJE \sim \triangle CBE$ with scale factor $\frac{PE}{CE} = \frac{2}{10} = \frac{1}{5}$. Therefore $PJ = \frac{1}{5}BC = \frac{8}{5}$ and $JE = \frac{1}{5}BE = \frac{6}{5}$. Furthermore, $JF = JE + EF = \frac{6}{5} + 2 = \frac{16}{5}$.

Note that $HF = PJ = \frac{8}{5}$ and $PH = JF = \frac{16}{5}$ because $PJFH$ is a rectangle. Also note that $HQ = QF - HF = 8 - \frac{8}{5} = \frac{32}{5}$. Finally, $PQ^2 = PH^2 + HQ^2 = \left(\frac{16}{5}\right)^2 + \left(\frac{32}{5}\right)^2 = \left(\frac{16}{5}\right)^2 (1^2 + 2^2) = \left(\frac{16}{5}\right)^2 \cdot 5$. Therefore $PQ = \sqrt{\left(\frac{16}{5}\right)^2 \cdot 5} = \frac{16}{5}\sqrt{5} = \boxed{\frac{16\sqrt{5}}{5}}$.



Round 4 - Logs, Exponents, and Radicals

1. Find n given that n is an integer and

$$n < \log_2 25 < n + 1$$

Solution: Let $x = \log_2 25$. Note that if $n < x < n + 1$, then:

$$2^n < 2^x < 2^{n+1}$$

because the function $f(x) = 2^x$ increases as x increases for all x . In other words, if $n < x$ then $2^n < 2^x$, and likewise for x and $n + 1$. Recall the inverse identity between logarithms and exponents: if $\log_2 25 = x$ then $2^x = 25$. Replace 2^x with 25 in the dual inequality above:

$$2^n < 25 < 2^{n+1}$$

Finally, consider the powers of 2: $2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32$. Then $16 < 25 < 32$, or $2^4 < 25 < 2^5$, and $n = \boxed{4}$ satisfies the inequality.

2. Find x :

$$\left(\frac{1}{81}\right)^{2x-1} = (243)^{x+7}$$

Solution: First, note that $81 = 9^2 = (3^2)^2 = 3^4$ and $243 = 3 \cdot 81 = 3 \cdot 3^4 = 3^5$. Replace 81 and 243 with powers of 3 in the given equation and apply laws of exponents to convert each side of the equation to a power of 3:

$$\begin{aligned} \left(\frac{1}{81}\right)^{2x-1} &= (243)^{x+7} \\ \left(\frac{1}{3^4}\right)^{2x-1} &= (3^5)^{x+7} \\ (3^{-4})^{2x-1} &= 3^{5(x+7)} \\ 3^{-4(2x-1)} &= 3^{5x+35} \end{aligned}$$

after applying the laws of exponents $\left(\frac{1}{b}\right)^x = b^{-x}$ and $(b^x)^y = b^{(x \cdot y)}$. Note that the last equation is in the form $3^{f(x)} = 3^{g(x)}$. Equate the exponents: $f(x) = g(x)$, or $-4(2x - 1) = 5x + 35$. Solve for x : $-8x + 4 = 5x + 35$; $4 - 35 = 5x + 8x$; $-31 = 13x$, and $x = \boxed{-\frac{31}{13}}$.

3. Find x if

$$x = \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}$$

Solution: Start by squaring both sides of the given equation, simplify, and isolate the radical on one side of the equation:

$$\begin{aligned} x^2 &= \left(\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} \right)^2 \\ x^2 &= \left(\sqrt{x - \frac{1}{x}} \right)^2 + \left(\sqrt{1 - \frac{1}{x}} \right)^2 + 2\sqrt{x - \frac{1}{x}} \cdot \sqrt{1 - \frac{1}{x}} \\ x^2 &= x - \frac{1}{x} + 1 - \frac{1}{x} + 2\sqrt{\left(x - \frac{1}{x}\right) \left(1 - \frac{1}{x}\right)} \\ x^2 - x - 1 + \frac{2}{x} &= 2\sqrt{x - 1 - \frac{1}{x} + \left(-\frac{1}{x}\right)^2} \end{aligned}$$

after applying the relations $(a+b)^2 = a^2 + b^2 + 2ab$, $(\sqrt{a})^2 = a$, $(a-c)(b-c) = ab - bc - ac + c^2$ and subtracting $x + 1 - \frac{2}{x}$ from both sides of the equation. Note the similarity between the expression on the left hand side of the equation and the expression under the radical. This similarity is more striking after combining terms with a common denominator both on the left hand side and under the radical:

$$\begin{aligned} \frac{x^3 - x^2 - x + 2}{x} &= 2\sqrt{\frac{x^3 - x^2 - x + 1}{x^2}} \\ \frac{x^3 - x^2 - x + 1 + 1}{x} &= 2\frac{\sqrt{x^3 - x^2 - x + 1}}{x} \\ x^3 - x^2 - x + 1 + 1 &= 2\sqrt{x^3 - x^2 - x + 1} \end{aligned}$$

Next, let $y = \sqrt{x^3 - x^2 - x + 1}$, so that $y^2 = x^3 - x^2 - x + 1$. Substitute these into the above equation and $y^2 + 1 = 2y$, or $y^2 - 2y + 1 = 0$, or $(y - 1)^2 = 0$. Then $y = 1$, $y^2 = 1 = x^3 - x^2 - x + 1$, and

$$\begin{aligned} x^3 - x^2 - x + 1 &= 1 \\ x^3 - x^2 - x &= x(x^2 - x - 1) = 0 \end{aligned}$$

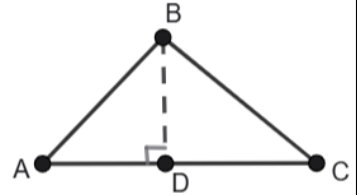
Ignore the extraneous solution $x = 0$ and apply the quadratic formula to $x^2 - x - 1 = 0$: $x = \frac{1 \pm \sqrt{(-1)^2 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2}$. Note that x is the sum of two square roots, so $x > 0$, and the negative solution

is extraneous. Therefore $x = \boxed{\frac{1 + \sqrt{5}}{2}}$.

Round 5 - Trigonometry

1. Find the area of $\triangle ABC$ if $m\angle A = 45^\circ$, $AB = 6$, and $AC = 10$.

Solution: Draw triangle $\triangle ABC$. Construct altitude $\overline{BD}$ perpendicular to $\overline{AC}$. Let $h = BD$ (the height of $\triangle ABC$). Recall from right triangle trigonometry: $\text{sine} = \frac{\text{opposite}}{\text{hypotenuse}}$. Note that $\overline{BD}$ is opposite $\angle A$ in right triangle $\triangle ABD$, and that $\overline{AB}$ is the hypotenuse. Therefore $\sin \angle A = \frac{BD}{AB}$ and



$$BD = (AB) \sin \angle A = 6 \sin 45^\circ = 6 \frac{\sqrt{2}}{2} = 3\sqrt{2}$$

after recalling that $\sin 45^\circ = \frac{\sqrt{2}}{2}$. Finally, recall that the area of a triangle is equal to half the product of its base b and its height h . In this case $b = AC = 10$, $h = BD = 3\sqrt{2}$, and

$$\begin{aligned} [\triangle ABC] &= \frac{1}{2} \cdot 10 \cdot 3\sqrt{2} \\ &= 5 \cdot 3\sqrt{2} = \boxed{15\sqrt{2}} \end{aligned}$$

Alternative Solution: Recall the formula for the area of a triangle:

$$[\triangle ABC] = \frac{1}{2}bc \sin \angle A$$

where $[\triangle ABC]$ is the area of $\triangle ABC$, $b = AC$ ($\overline{AC}$ is the side opposite $\angle B$), and $c = AB$ ($\overline{AB}$ is the side opposite $\angle C$). Therefore

$$\begin{aligned} [\triangle ABC] &= \frac{1}{2} \cdot 10 \cdot (6) \sin 45^\circ \\ &= 5 \cdot 6 \left(\frac{\sqrt{2}}{2} \right) = \frac{30\sqrt{2}}{2} = \boxed{15\sqrt{2}} \end{aligned}$$

2. Let $\frac{\frac{1}{\cos x}}{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}} = a \sin^b x \cos^c x$. Find the ordered triple (a, b, c) .

Solution: Start by adding the two fractions in denominator of the left hand side: $\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} =$

$$\frac{\cos^2 x}{\sin x \cos x} + \frac{\sin^2 x}{\sin x \cos x} = \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x}. \text{ Then } \frac{\frac{1}{\cos x}}{\frac{1}{\sin x \cos x}} = \frac{\sin x \cos x}{\cos x} = \sin x = 1 \cdot \sin^1 x \cos^0 x.$$

Therefore $(a, b, c) = \boxed{(1, 1, 0)}$.

3. Let y^* be the maximum value of $f(\theta)$ and let θ^* be the smallest positive value of θ such that $f(\theta) = y^*$ where:

$$f(\theta) = 2\sqrt{3}\sin(\theta + 45^\circ) + 2\sin(\theta - 45^\circ).$$

Find the ordered pair (θ^*, y^*) where θ^* is measured in degrees.

Solution: Let $\phi = \theta + 45^\circ$. Then $\theta = \phi - 45^\circ$ and $f(\theta)$ can be rewritten as a function of ϕ :

$$g(\phi) = 2\sqrt{3}\sin\phi + 2\sin(\phi - 90^\circ)$$

Note that $g(\phi)$ is equal to $f(\theta)$ shifted by 45° . Therefore $g(\phi)$ and $f(\theta)$ have the same maximum value y^* and the angles that yield that maximum are related by $\theta^* = \phi^* - 45^\circ$.

Recall the formula for the sine of a sum of angles, specifically $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$. Then $\sin(\phi - 90^\circ) = \sin\phi\cos(-90^\circ) + \sin(-90^\circ)\cos\phi = (\sin\phi) \cdot 0 + (-1) \cdot \cos\phi = -\cos\phi$. Substitute this function into the equation for $g(\phi)$:

$$g(\phi) = 2\sqrt{3}\sin\phi - 2\cos\phi$$

Note that $\sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{(12 + 4)} = \sqrt{16} = 4$. Next, factor 4 from $g(\phi)$:

$$g(\phi) = 4 \left(\frac{2\sqrt{3}}{4}\sin\phi - \frac{2}{4}\cos\phi \right) = 4 \left(\frac{\sqrt{3}}{2}\sin\phi + -\frac{1}{2}\cos\phi \right)$$

Compare this function inside the parentheses to the sine of a sum of angles formula. Find γ such that $\cos(\gamma) = \frac{\sqrt{3}}{2}$ and $\sin(\gamma) = -\frac{1}{2}$ by inspecting the unit circle. Then $\gamma = -30^\circ$ and:

$$\begin{aligned} g(\phi) &= 4(\cos(-30^\circ)\sin\phi + \sin(-30^\circ)\cos\phi) \\ &= 4\sin(\phi - 30^\circ) \end{aligned}$$

where the formula for the sine of a sum of angles has been applied in reverse in the last step. Now the maximum value of $\sin x$ is 1, so the maximum value of $g(\phi)$ is 4, which is also the maximum value of $f(\theta)$. The maximum value of $\sin x$ is reached when $x = 90^\circ$. Therefore $\phi^* - 30^\circ = 90^\circ$ and $\phi^* = 120^\circ$. Then $\theta^* = \phi^* - 45^\circ = 120^\circ - 45^\circ = 75^\circ$. In summary, $(\theta^*, y^*) = \boxed{(75^\circ, 4)}$.

Team Round

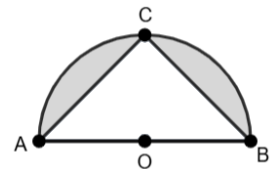
- Let N equal the product of the first 100 odd prime numbers. Find the least significant (rightmost) digit of N , as written in base 10.

Solution: Recall that the product of 5 and any odd number will end in 5, that is, its rightmost digit is 5. Note that $N \div 5$, the product of the first 100 odd prime numbers other than 5, will be odd because the product of any number of odd numbers is odd. Therefore the rightmost digit of $N = 5 \cdot (N \div 5)$ is $\boxed{5}$.

- A battalion of 880 soldiers is made up of four companies of equal size. Each of those companies is made up of five platoons of equal size. Each of those platoons is made up of four squads of equal size. Find the number of soldiers in a squad.

Solution: Each of the four companies has $880 \div 4 = 220$ soldiers. Each of the five platoons in each company has $220 \div 5 = 44$ soldiers. Each of the four squads in each platoon has $44 \div 4 = \boxed{11}$ soldiers.

- Isosceles $\triangle ABC$ is inscribed in semicircle O as shown in the figure at right. The radius of the semicircle $OA = 10$. Find the exact area of the shaded region. Do not approximate π .



Solution: Let A_s be the area of semicircle O . Let A_t be the area of $\triangle ABC$. Observe that the shaded area is equal to $A_s - A_t$.

Recall that the area of a circle is πr^2 , where r is the radius of the circle, and that the area of a semicircle is half that, or $\frac{\pi r^2}{2}$. It was given that $r = 10$. Then $A_s = \pi \frac{10^2}{2} = \pi \frac{100}{2} = 50\pi$.

Note that $\overline{OC}$ is the median of isosceles $\triangle ABC$ because $\overline{OA}$ and $\overline{OB}$ are radii and therefore congruent. Recall that the median of an isosceles triangle divides the triangle into congruent right triangles, which can be proved by SSS congruence. Then $\overline{OC} \perp \overline{AB}$ and $\overline{OC}$ is an altitude of $\triangle ABC$. Also, its length $OC = 10$ because it is a radius of semicircle O .

Recall that the area of a triangle is equal to $\frac{1}{2}bh$, where b is the length of the base and h is the height, that is, the length of the altitude of the triangle. In this case, $b = AB = 20$ and $h = OC = 10$, so $A_t = \frac{1}{2} \cdot 20 \cdot 10 = 100$. Finally, the shaded area is equal to $A_s - A_t = \boxed{50\pi - 100}$.

4. Given that $\log_{\sqrt{n}} m^3 = \sqrt{2}$, find:

$$[\log_m (n^2 \cdot m)]^2$$

Solution: Convert the lefthand side of the given equation into a base m logarithm, starting by applying the identity related to the logarithm of an exponential function and continuing after recalling the following variation on the chain rule for logarithms $\log_b a = \frac{1}{\log_a b}$:

$$\begin{aligned} \log_{\sqrt{n}} m^3 &= 3 \log_{\sqrt{n}} m \\ &= 3 \cdot \frac{1}{\log_m \sqrt{n}} = \frac{3}{\log_m \sqrt{n}} \\ &= \frac{3}{\log_m n^{\frac{1}{2}}} = \frac{3}{\frac{1}{2} \log_m n} = \frac{2 \cdot 3}{\log_m n} \\ &= \frac{6}{\log_m n} \end{aligned}$$

after recalling and applying the equivalence $\sqrt{n} = n^{\frac{1}{2}}$ and the logarithm of exponent identity. Then

$$\frac{6}{\log_m n} = \sqrt{2}$$

and $\log_m n = \frac{6}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$.

Next simplify $\log_m n^2 \cdot m$ by applying the log of a product and log of an exponential identities and substituting the value $\log_m n = 3\sqrt{2}$:

$$\log_m (n^2 \cdot m) = \log_m n^2 + \log_m m = 2 \log_m n + 1 = 2 \cdot 3\sqrt{2} + 1 = 6\sqrt{2} + 1$$

Finally, square and simplify this last expression: $(6\sqrt{2} + 1)^2 = (6\sqrt{2})^2 + 2 \cdot 6\sqrt{2}(1) + 1^2 = 72 + 12\sqrt{2} + 1 = \boxed{73 + 12\sqrt{2}}$.

5. In triangle $\triangle ABC$, $\tan \angle A = 3$ and $\tan \angle B = 4$. Find the value of $\tan \angle C$.

Solution: Recall that the measures of the angles of a triangle sum to 180° . Then $m\angle A + m\angle B + m\angle C = 180^\circ$ and $m\angle C = 180^\circ - m\angle A - m\angle B$. Therefore $\tan \angle C = \tan (180^\circ - m\angle A - m\angle B)$. Note that $\tan(180^\circ - x) = \frac{\sin(180^\circ - x)}{\cos(180^\circ - x)} = \frac{-\sin x}{\cos x} = -\tan x$. Therefore $\tan \angle C = -\tan(A + B)$.

Next, recall that $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$. Therefore

$$\tan \angle C = -\frac{\tan A + \tan B}{1 - (\tan A)(\tan B)} = -\frac{3 + 4}{1 - 3 \cdot 4} = -\frac{7}{1 - 12} = \frac{-7}{-11} = \boxed{\frac{7}{11}}.$$

6. The product of a set of distinct positive integers is 126. What is the least possible sum of this set of integers?

Solution: First find the prime factorization of 126: $126 = 2 \cdot 63 = 2 \cdot 9 \cdot 7 = 2 \cdot 3^2 \cdot 7$. Note that $2 \cdot 3 \cdot 3 \cdot 7 = 126$ and $2 + 3 + 3 + 7 = 15$.

Note that this set $\{2, 3, 3, 7\}$ is not distinct, and one of the 3's in this set must be combined with another element in a product. Combine 3 with the element or elements such that the product will increase the sum the least. Note that there is no product of two elements that will not increase the sum; only $2 \cdot 2 = 2 + 2$, but there is only one factor of 2 in the set. Next, note that $2 \cdot 3 = 6 = (2 + 3) + 1$, and the sum is increased by 1, the least possible increase. Then $2 \cdot 3 + 3 + 7 = 6 + 3 + 7 = \boxed{16}$.

7. Given that $x + y = 4$ and $x + z = 9$, find the value of $y - z$.

Solution: Subtract the second equation from the first to find the answer:

$$\begin{array}{r} x + y = 4 \\ - (x + z = 9) \\ \hline y - z = \boxed{-5} \end{array}$$

8. Given that $(a + bi)^2 = 3 + 4i$, where a and b are Real numbers and $i^2 = -1$, find all possible ordered pairs (a, b) .

Solution: Expand $(a + bi)^2$ and put it into the form $p + qi$:

$$\begin{aligned}(a + bi)^2 &= a^2 + 2abi + (bi)^2 \\ &= a^2 + 2abi + b^2(i^2) \\ &= a^2 + 2abi + b^2(-1) = a^2 - b^2 + 2abi\end{aligned}$$

Equate the Real part of the last expression to 3 and the Imaginary part to 4 to create the system:

$$\begin{aligned}a^2 - b^2 &= 3 \\ 2ab &= 4, \text{ or } ab = 2\end{aligned}$$

Then $b = \frac{2}{a}$. Substitute this value for b in the top equation and $a^2 - \frac{2^2}{a^2} = 3$. Multiply this equation by a^2 and subtract $3a^2$ from both sides, then factor the polynomial in a^2 :

$$\begin{aligned}a^4 - 3a^2 - 4 &= 0 \\ (a^2 - 4)(a^2 + 1) &= 0\end{aligned}$$

so that $a^2 - 4 = 0$ or $a^2 + 1 = 0$. Solve the first equation: $a = \pm 2$. Note that there is no real solution to $a^2 + 1 = 0$. Because $b = \frac{2}{a}$, if $a = 2$ then $b = 1$, and if $a = -2$, $b = -1$. The possible (a, b) pairs are therefore $\boxed{(2, 1) \text{ and } (-2, -1)}$ and no others.

9. Given the following expression:

$$\sin [2 \cos^{-1}(-x)]$$

find an equivalent expression in terms of x that has no trigonometric functions.

Solution: Apply the double angle identity for sine ($\sin 2x = 2 \sin x \cos x$):

$$\sin [2 \cos^{-1}(-x)] = 2 \sin[\cos^{-1}(-x)] \cos[\cos^{-1}(-x)].$$

Apply the definition of an inverse function and $\cos[\cos^{-1}(-x)] = -x$.

Next, recall the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$. Let $\theta = \cos^{-1}(-x)$, so that $\cos \theta = -x$ and $\cos^2 \theta = x^2$. Then $\sin^2 \theta = 1 - \cos^2 \theta = 1 - x^2$ and $\sin \theta = \sin[\cos^{-1}(-x)] = \pm \sqrt{1 - x^2}$.

Note that the range of $\cos^{-1} x$ is $[0, \pi)$ so that $\sin(\cos^{-1} x)$ will never be negative; in this case, $\sin[\cos^{-1}(-x)] = \sqrt{1 - x^2}$. Substitute the expressions for the sin and cos terms into the highlighted equation and $\sin [2 \cos^{-1}(-x)] = \boxed{-2x\sqrt{1 - x^2}}$.